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The asymmetric flux tube

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Abstract

We consider a relativistic flux tube with arbitrary mass spinless quarks at the ends. In the unequal mass case the additional constraint that the center of momentum is fixed must be included. An analytic classical circular solution is found and is used as a correspondence limit. The quantized equations for mesons at rest are constructed and solved. A comparison is made with all available spin-averaged heavy-light and heavy-heavy states. We find evidence for running of the heavy quark masses as well as the short range coupling constant.

1 Introduction

The relativistic flux tube (RFT) model shows promise to provide a realistic description of all meson states. The RFT model is in essence a description of dynamical confinement [1]-[4]. For slowly moving quarks rigorous QCD relativistic corrections [5]-[7] clearly demonstrate that the scalar confinement potential picture is incorrect [2, 7]. On the other hand, the RFT dynamics are consistent with both spin-dependent [3, 8, 9] and spin-independent [2, 3] QCD expectations. The basic assumption of the RFT model is that the QCD dynamical ground state for large quark separation consists of a rigid straight tubelike color flux configuration connecting the quarks. In this idealized limit the quarks and tube are shown in Fig. 1. Although based on a simple physical picture this model follows directly from the QCD Lagrangian with natural approximations to the Wilson action [10].

The heavy-light mesons are important for many reasons. For our present purposes, these mesons exhibit relativistic dynamics while still maintaining some simplifying aspects. The heavy quark mass suppresses most spin dependence so that a spinless quark analysis has considerable validity. The one heavy quark also means the reduced Salpeter equation [11] will be appropriate and that a relative time degree of freedom is unimportant.

In the present work we have focused on the following points:

1. The extension to unequal quark masses requires the solution of several technical problems. These problems have their origin in locating the center of momentum and ensuring that the total momentum of the meson vanishes.
2. The classical solution for rotational motion with unequal quark masses provides a useful check on our quantum solution. We have obtained an analytic solution which satisfies the requirement that the total meson momentum is zero.

3. Both spinless and fermionic quark analyses share a common orbital angular momentum analysis. The techniques developed here will therefore be of direct utility in the more realistic calculation.
4. Because of the suppressed spin dependence due to the small color magnetic moment of the heavy quark, a spin-averaged analysis is realistic.
5. We reconsider the question of the symmetrization of operators in the quantized RFT equations. Algorithms are developed for finding the symmetrical perpendicular velocity operators.
6. We have considered all the spin averaged heavy-heavy and heavy-light states and found the expected *running* of the heavy quark masses, as well as the short range interaction, follows in a natural way.

As established previously [1]-[4] the classical angular momentum of two quarks plus two tube segments joined at the CM is

$$J = W_{r_1} \gamma_{\perp 1} v_{\perp 1} r_1 + 2ar_1^2 f(v_{\perp 1}) + (1 \rightarrow 2) , \quad (1)$$

where

$$W_{r_i} = \sqrt{p_{r_i}^2 + m_i^2} , \quad (2)$$

$$4v_{\perp} f(v_{\perp}) = \frac{\arcsin v_{\perp}}{v_{\perp}} - \gamma_{\perp}^{-1} , \quad (3)$$

$$\gamma_{\perp}^{-2} = 1 - v_{\perp}^2 . \quad (4)$$

The classical Hamiltonian for this system is

$$H = W_{r_1} \gamma_{\perp 1} + ar_1 \frac{\arcsin v_{\perp 1}}{v_{\perp 1}} + (1 \rightarrow 2) . \quad (5)$$

For unequal mass quarks additional conditions [3] must be imposed so that the meson CM is at rest. These conditions are

$$p_{r_1} = p_{r_2} \equiv p_r , \quad (6)$$

$$P_{\perp} = 0 = W_{r_1} \gamma_{\perp_1} v_{\perp_1} + a \frac{r_1}{v_{\perp_1}} (1 - \gamma_{\perp_1}^{-1}) - (1 \rightarrow 2) . \quad (7)$$

Finally, the straight flux tube condition is

$$\frac{v_{\perp_1}}{r_1} = \frac{v_{\perp_2}}{r_2} . \quad (8)$$

For equal mass quarks the P_{\perp} condition (7) is satisfied since $v_{\perp_1} = v_{\perp_2}$. For heavy-light mesons where $m_2 \gg m_1$ and also $m_2 \gg ar_1$, the P_{\perp} condition can be satisfied with $v_{\perp_2} \simeq 0$ (i.e., the CM point is coincident with the heavy quark). For all other (asymmetrical) mesons all of the above relations (1-8) must be used in the construction of the solution. This aspect is the central problem addressed here.

In section 2 we consider the purely rotational classical solutions to the RFT equations. These results will provide useful correspondence check on the quantum solution. Section 3 formulates the quantized RFT equations and their solutions are discussed in section 4. In section 5 we test the model by comparing to all observed spin-averaged masses of mesons containing one or two heavy quarks. We observe that better global agreement with the data is obtained if the heavy quark masses run with QCD scale. Our conclusions are summarized in section 6.

2 Classical (Yrast) Solution

The classical solution of the RFT equations having minimum energy for a given angular momentum (the *yrast* solution) corresponds to circular motion, i.e., $p_r = 0$. In this case the RFT equations (1) and (5) become

$$J = m_1 \gamma_{\perp_1} v_{\perp_1} r_1 + 2ar_1^2 f(v_{\perp_1}) + (1 \rightarrow 2) , \quad (9)$$

$$H = m_1 \gamma_{\perp_1} + ar_1 \frac{\arcsin v_{\perp_1}}{v_{\perp_1}} + (1 \rightarrow 2) . \quad (10)$$

The condition (7) that the CM momentum vanishes becomes

$$P_{\perp} = 0 = m_1 \gamma_{\perp_1} v_{\perp_1} + a \frac{r_1}{v_{\perp_1}} (1 - \gamma_{\perp_1}^{-1}) - (1 \rightarrow 2) . \quad (11)$$

Along with the straight tube condition (8), the above three relations can be used to find the yrast solution $M = H(J)$.

A short cut to the yrast solution directly uses the circular dynamical condition. In the Lagrangian approach [1, 12] the radial derivative of the Lagrangian, must vanish since $\dot{p}_r = 0$. Another way to obtain the same result is to consider the force on the i^{th} quark in two reference systems. In its rest frame the quark experiences a force $-a\hat{\mathbf{r}}_i$ due to the tube. In the CM rest frame, where the quark is moving with the velocity v_\perp perpendicular to the tube, the transverse force is a/γ_{\perp_i} and for circular orbite this must equal the mass $m_i\gamma_{\perp_i}$ times the centripetal acceleration or

$$(m_i\gamma_{\perp_i})\frac{v_{\perp_i}^2}{r_i} = \frac{a}{\gamma_{\perp_i}} . \quad (12)$$

This, along with the straight string condition (8), assures that the meson is at rest (i.e., the total linear momentum (11) vanishes). From (12) we also have

$$v_{\perp_i}^2 = \frac{ar_i}{m_i + ar_i} , \quad (13)$$

$$\gamma_{\perp_i}^2 = 1 + \frac{ar_i}{m_i} . \quad (14)$$

The total interquark distance is evidently

$$r = r_1 + r_2 , \quad (15)$$

and using (8) and (13), we obtain

$$r_1 = \frac{m_2 + ar}{m_1 + m_2 + 2ar} , \quad (16)$$

$$r_2 = \frac{m_1 + ar}{m_1 + m_2 + 2ar} . \quad (17)$$

When the above expressions are substituted back into (13) and (14), we find

$$v_{\perp_1}^2 = \frac{ar(m_2 + ar)}{(m_1 + ar)(m_1 + m_2 + ar)} , \quad (18)$$

$$v_{\perp_2}^2 = \frac{ar(m_1 + ar)}{(m_2 + ar)(m_1 + m_2 + ar)} , \quad (19)$$

$$\gamma_{\perp_i}^2 = \frac{(m_i + ar)(m_1 + m_2 + ar)}{m_i(m_1 + m_2 + 2ar)} . \quad (20)$$

The quantities r_i , v_{\perp_i} and γ_{\perp_i} have now been expressed in terms of the interquark distance r . The circular RFT equations can similarly be expressed in terms of r . Equation (9) for the meson angular momentum becomes

$$\begin{aligned} 2\frac{J}{r} &= \frac{\sqrt{ar}}{(m_1 + m_2 + 2ar)^{\frac{3}{2}}} \left[\sqrt{m_1}(m_2 + ar)^{\frac{3}{2}} + \sqrt{m_2}(m_1 + ar)^{\frac{3}{2}} \right] \\ &+ \frac{(m_1 + m_2 + ar)(m_1 + ar)(m_2 + ar)}{(m_1 + m_2 + 2ar)^2} [\arcsin v_{\perp_1} + \arcsin v_{\perp_2}] , \end{aligned} \quad (21)$$

and the state mass M from (10) is

$$\begin{aligned} M &= \sqrt{\frac{m_1 + m_2 + ar}{m_1 + m_2 + 2ar}} \left[\sqrt{m_1(m_1 + ar)} + \sqrt{m_2(m_2 + ar)} \right] \\ &+ \frac{\sqrt{ar(m_1 + m_2 + ar)(m_1 + ar)(m_2 + ar)}}{m_1 + m_2 + 2ar} [\arcsin v_{\perp_1} + \arcsin v_{\perp_2}] . \end{aligned} \quad (22)$$

In (21) and (22) v_{\perp_1} and v_{\perp_2} are functions of r given by (18) and (19). The above circular solutions reduce to the equal mass case considered earlier [12].

Now we can easily establish three limiting cases:

1. Heavy-heavy case ($m_1, m_2 \gg ar$)

Equations (21) and (22) can be expanded in the small quantities $\frac{ar}{m_1}$ and $\frac{ar}{m_2}$ with results

$$J^2 = \left(\frac{m_1 m_2 a}{m_1 + m_2} \right) r^3 , \quad (23)$$

$$M = m_1 + m_2 + \frac{3}{2}ar . \quad (24)$$

From the above the shifted Regge slope is

$$\frac{dJ}{d(M - m_1 - m_2)^2} = \left(\frac{m_1 m_2}{m_1 + m_2} \right)^{\frac{2}{3}} \frac{J^{-\frac{1}{3}}}{3a^{\frac{4}{3}}} . \quad (25)$$

2. Heavy-light case ($m_1 = 0, m_2 \gg ar$)

Again, from (21) and (22) in this limit we find

$$J = \frac{\pi ar^2}{4} , \quad (26)$$

$$M = m_2 + \frac{\pi ar}{2} . \quad (27)$$

Eliminating r the shifted Regge slope is

$$\frac{J}{(M - m_2)^2} = \frac{1}{\pi a} . \quad (28)$$

3. Light-light case ($m_1 = m_2 = 0$)

For zero mass quarks the tube carries all the rotational momentum and energy of the meson. In this limit we obtain

$$J = \frac{\pi ar^2}{8} , \quad (29)$$

$$M = \frac{\pi ar}{2} . \quad (30)$$

giving a Regge slope (Nambu) of

$$\frac{J}{M^2} = \frac{1}{2\pi a} . \quad (31)$$

The various limiting situations are illustrated in Figs. 2 and 3. We choose $m_2 = 1.5GeV$, $a = (2\pi)^{-1}GeV^2$ and plot the shifted Regge slope as a function of J . In Fig. 2 we take $m_1 = 0$. The solid curve shows a shifted slope of 2 units at small J as expected from the heavy-light limit of (28). At large J where $m_2 < ar$ the slope approaches the Nambu limit (31). For $m_1 = 0.5GeV$ the solid curve of Fig. 3

illustrates the yrast shifted Regge slope. At small J the heavy-heavy limit (25) is obtained, and for large J again the Nambu limit of unit slope is obtained. In both Figs. 2 and 3 the dashed curve is the quantum solution which will be discussed in section 4. It should be noted that in each case the Regge behaviour is most simply expressed in terms of excitation energy $M - m_1 - m_2$.

3 Quantization of the Classical Equations

3.1 Equal-mass and the heavy-light case

The classical equations (1-8) for the equal-mass case reduce to

$$\frac{J}{r} = W_r \gamma_\perp v_\perp + ar f(v_\perp) , \quad (32)$$

$$H = 2W_r \gamma_\perp + ar \frac{\arcsin v_\perp}{v_\perp} , \quad (33)$$

with

$$W_r = \sqrt{p_r^2 + m^2} \quad (34)$$

and

$$f(v_\perp) = \frac{1}{4v_\perp} \left(\frac{\arcsin v_\perp}{v_\perp} - \frac{1}{\gamma_\perp} \right) , \quad (35)$$

Since in the limit of large quark masses [1, 3]

$$v_{\perp i} \rightarrow \frac{J}{m_i r} , \quad (36)$$

we have a good reason to believe that $v_{\perp 1}$ and $v_{\perp 2}$ should be hermitian operators. Also, if we could invert the classical angular momentum equation for the equal-mass case (32), we would have

$$v_\perp = v_\perp(r, p_r^2, J) , \quad (37)$$

which means that $[v_\perp, r] \neq 0$ when v_\perp and r are considered as quantum-mechanical operators. Therefore, the classical equations should be symmetrized and quantized

as [1]

$$J \rightarrow \sqrt{l(l+1)}, \quad p_r^2 \rightarrow -\frac{1}{r} \frac{\partial^2}{\partial r^2} r, \quad (38)$$

$$2 \frac{\sqrt{l(l+1)}}{r} = \{W_r, \gamma_\perp v_\perp\} + a\{r, f(v_\perp)\}, \quad (39)$$

$$H = \{W_r, \gamma_\perp\} + \frac{a}{2} \left\{ r, \frac{\arcsin v_\perp}{v_\perp} \right\}, \quad (40)$$

where $\{A, B\} \equiv AB + BA$. Similarly, the heavy-light case classical equations with m_2 as the heavy quark mass,

$$\frac{J}{r} = W_r \gamma_\perp v_\perp + 2ar f(v_\perp), \quad (41)$$

$$H = m_2 + W_r \gamma_\perp + ar \frac{\arcsin v_\perp}{v_\perp}, \quad (42)$$

after quantization become

$$\frac{\sqrt{l(l+1)}}{r} = \frac{1}{2} \{W_r, \gamma_\perp v_\perp\} + a\{r, f(v_\perp)\}, \quad (43)$$

$$H = m_2 + \frac{1}{2} \{W_r, \gamma_\perp\} + \frac{a}{2} \left\{ r, \frac{\arcsin v_\perp}{v_\perp} \right\}. \quad (44)$$

3.2 General case

In the general case with $m_1 \neq m_2$ things get more complicated, but proceed along similar lines. Besides equation defining the angular momentum, we also require that total momentum of the system must be zero, which is trivially satisfied in the two special cases considered above.

We begin by using the straight tube condition (8) and the quark separation variable (15) which give

$$r_i = \frac{v_{\perp i}}{v_{\perp 1} + v_{\perp 2}} r, \quad i = 1, 2. \quad (45)$$

The classical RFT equations (1-7) can then be expressed in terms of v_\perp 's and r :

$$\begin{aligned} \frac{J}{r} &= W_{r_1} \frac{\gamma_{\perp_1} v_{\perp_1}^2}{v_{\perp_1} + v_{\perp_2}} + W_{r_2} \frac{\gamma_{\perp_2} v_{\perp_2}^2}{v_{\perp_1} + v_{\perp_2}} \\ &+ ar \frac{1}{(v_{\perp_1} + v_{\perp_2})^2} (\tilde{f}(v_{\perp_1}) + \tilde{f}(v_{\perp_2})) , \end{aligned} \quad (46)$$

$$\begin{aligned} P_\perp = 0 &= W_{r_1} \gamma_{\perp_1} v_{\perp_1} - W_{r_2} \gamma_{\perp_2} v_{\perp_2} \\ &- ar \frac{1}{v_{\perp_1} + v_{\perp_2}} \left(\frac{1}{\gamma_{\perp_1}} - \frac{1}{\gamma_{\perp_2}} \right) , \end{aligned} \quad (47)$$

$$\begin{aligned} H &= W_{r_1} \gamma_{\perp_1} + W_{r_2} \gamma_{\perp_2} \\ &+ ar \frac{1}{v_{\perp_1} + v_{\perp_2}} (\arcsin v_{\perp_1} + \arcsin v_{\perp_2}) . \end{aligned} \quad (48)$$

Here we have

$$W_{r_i} = \sqrt{p_r^2 + m_i^2} , \quad (49)$$

because of radial momentum conservation, and the function $\tilde{f}(v_\perp)$ is defined as

$$\tilde{f}(v_\perp) = \frac{1}{2} (\arcsin v_\perp - \frac{v_\perp}{\gamma_\perp}) = \frac{f(v_\perp)}{2v_\perp^2} . \quad (50)$$

If we could invert equations (46) and (47) for v_{\perp_1} and v_{\perp_2} , they would be in general different functions of r, p_r and J , so that corresponding quantum mechanical operators will not commute. Therefore, our quantized equations will contain products of three non-commuting factors, and we have to find the way to symmetrize them. In doing that, we have to keep in mind that our procedure must reduce to equations (39,40) and (43,44) in the equal-mass and the heavy-light limits, respectively. The easiest way to ensure this is to symmetrize first between non-commuting factors containing v_{\perp_1} and v_{\perp_2} to obtain symmetric operators $\hat{\mathcal{O}}(v_{\perp_1}, v_{\perp_2})$, and then to symmetrize between these operators and radial operators. If we do that, our quantized equations (46-48) become

$$\frac{\sqrt{l(l+1)}}{r} = \frac{1}{2} \{ W_{r_1}, \frac{1}{2} \{ \gamma_{\perp_1} v_{\perp_1}^2, \frac{1}{v_{\perp_1} + v_{\perp_2}} \} \}$$

$$\begin{aligned}
& + \frac{1}{2} \{W_{r_2}, \frac{1}{2} \{ \gamma_{\perp_2} v_{\perp_2}^2, \frac{1}{v_{\perp_1} + v_{\perp_2}} \} \} \\
& + \frac{1}{2} a \{ r, \frac{1}{2} \{ \tilde{f}(v_{\perp_1}) + \tilde{f}(v_{\perp_2}), \frac{1}{(v_{\perp_1} + v_{\perp_2})^2} \} \} , \tag{51}
\end{aligned}$$

$$\begin{aligned}
P_{\perp} = 0 & = \frac{1}{2} \{W_{r_1}, \gamma_{\perp_1} v_{\perp_1}\} - \frac{1}{2} \{W_{r_2}, \gamma_{\perp_2} v_{\perp_2}\} \\
& - \frac{a}{2} \{ r, \frac{1}{2} \{ \frac{1}{v_{\perp_1} + v_{\perp_2}}, \frac{1}{\gamma_{\perp_1}} - \frac{1}{\gamma_{\perp_2}} \} \} , \tag{52}
\end{aligned}$$

$$\begin{aligned}
H & = \frac{1}{2} \{W_{r_1}, \gamma_{\perp_1}\} + \frac{1}{2} \{W_{r_2}, \gamma_{\perp_2}\} \\
& + \frac{a}{2} \{ r, \frac{1}{2} \{ \frac{1}{v_{\perp_1} + v_{\perp_2}}, \arcsin v_{\perp_1} + \arcsin v_{\perp_2} \} \} . \tag{53}
\end{aligned}$$

4 Numerical Solution of the Quantized Equations

The RFT Hamiltonian (53) contains two unknown operators, v_{\perp_1} and v_{\perp_2} , which are in turn defined by equations (51) and (52). If we introduce a complete set of basis states $\{e_k(\mathbf{r})\}$ and then truncate at a finite number N [13],

$$\psi(\mathbf{r}) \simeq \sum_{k=1}^N c_k e_k(\mathbf{r}) , \tag{54}$$

these equations become two coupled transcendental $N \times N$ matrix equations involving unknown matrices of v_{\perp_1} and v_{\perp_2} , and known r , $\frac{1}{r}$, and W_{r_i} matrices [4]. Our numerical solution for v_{\perp_1} and v_{\perp_2} is based on a simple $x = F(x)$ iteration algorithm applied to transcendental matrix equations.

Equation (51) can be written in the form

$$\begin{aligned}
W_{r_1} \frac{1}{v_{\perp_1} + v_{\perp_2}} \gamma_{\perp_1} v_{\perp_1}^2 & = 4 \frac{\sqrt{l(l+1)}}{r} - W_{r_1} \gamma_{\perp_1} v_{\perp_1}^2 \frac{1}{v_{\perp_1} + v_{\perp_2}} \\
& - \{ \gamma_{\perp_1} v_{\perp_1}^2, \frac{1}{v_{\perp_1} + v_{\perp_2}} \} W_{r_1}
\end{aligned}$$

$$\begin{aligned}
& - \{W_{r_2}, \{\gamma_{\perp_2} v_{\perp_2}^2, \frac{1}{v_{\perp_1} + v_{\perp_2}}\}\} \\
& - a\{r, \{\tilde{f}(v_{\perp_1}) + \tilde{f}(v_{\perp_2}), \frac{1}{(v_{\perp_1} + v_{\perp_2})^2}\}\} . \quad (55)
\end{aligned}$$

If we multiply this equation by $v_{\perp_1}^{-1} \gamma_{\perp_1}^{-1} (v_{\perp_1} + v_{\perp_2}) W_{r_1}^{-1}$ from the left, we get

$$\begin{aligned}
v_{\perp_1} &= v_{\perp_1}^{-1} \gamma_{\perp_1}^{-1} (v_{\perp_1} + v_{\perp_2}) W_{r_1}^{-1} \left(4 \frac{\sqrt{l(l+1)}}{r} - W_{r_1} \gamma_{\perp_1} v_{\perp_1}^2 \frac{1}{v_{\perp_1} + v_{\perp_2}} \right. \\
& - \{ \gamma_{\perp_1} v_{\perp_1}^2, \frac{1}{v_{\perp_1} + v_{\perp_2}} \} W_{r_1} \\
& - \{ W_{r_2}, \{ \gamma_{\perp_2} v_{\perp_2}^2, \frac{1}{v_{\perp_1} + v_{\perp_2}} \} \} \\
& \left. - a\{r, \{ \tilde{f}(v_{\perp_1}) + \tilde{f}(v_{\perp_2}), \frac{1}{(v_{\perp_1} + v_{\perp_2})^2} \} \} \right) . \quad (56)
\end{aligned}$$

We can do the same thing with equation (52) and obtain

$$\begin{aligned}
v_{\perp_1} &= \gamma_{\perp_1}^{-1} W_{r_1}^{-1} (-\gamma_{\perp_1} v_{\perp_1} W_{r_1} + \{W_{r_2}, \gamma_{\perp_2} v_{\perp_2}\} \\
& + a\{r, \frac{1}{2} \{ \frac{1}{v_{\perp_1} + v_{\perp_2}}, \frac{1}{\gamma_{\perp_1}} - \frac{1}{\gamma_{\perp_2}} \} \}) . \quad (57)
\end{aligned}$$

For a succesful iterative solution we must start with a good initial guess for matrices of v_{\perp_1} and v_{\perp_2} . We then have an initial guesses for matrices of all functions of these two operators and we can evaluate the right sides of equations (56) and (57) as the new guesses for v_{\perp_1} matrix (let us call them $v_{\perp_1}(J)$ and $v_{\perp_1}(P_{\perp})$. Since we want to have both equations (51) and (52) satisfied simultaneously, we mix these two guesses with weights depending on the extent to which the equations (51) and (52) are satisfied, with the initial guesses for v_{\perp_1} and v_{\perp_2} . For example, if the equation (51) is satisfied twice as well as the equation (52), then for the new guess for v_{\perp_1} we take $\frac{1}{3}v_{\perp_1}(J) + \frac{2}{3}v_{\perp_1}(P_{\perp})$. This ensures that both equations defining v_{\perp_1} and v_{\perp_2} will be simultaneously satisfied. Finally, since we are solving for matrices here, it is clear that our iteration scheme will be only marginally stable, so we employ a relaxation procedure. For the final new guess for v_{\perp_1} we take $(1 - \eta)v_{\perp_1}(old) + \eta v_{\perp_1}(new)$,

where η is the small number (usually $\eta \leq 0.1$, and it becomes smaller if we increase number N of the basis states that we are working with). After we have found the new guess for $v_{\perp 1}$, we do the same thing for $v_{\perp 2}$, and keep iterating until we achieve the required precision.

Using the same procedure, we were able to solve for v_{\perp} the equal-mass and the heavy-light case equations (39,43) even for as many as 50 basis states without too much effort and to achieve the very high accuracy of 8 decimal places. The initial guess in these two cases is obtained by finding the v_{\perp} from the non-symmetrized angular momentum equations [4], and then symmetrizing it by taking the average of the non-symmetric v_{\perp} matrix and its transpose. The eigenvalues obtained from the symmetrized Hamiltonians (40,44) are usually lower by at most a few MeV's than the eigenvalues obtained from the non-symmetrized equations. In Fig.4 we compare the non-symmetrized solution [4] for $m_1 = m_2 = 0$ (solid curve) with the symmetrized solution shown at integral angular momentum quantum number l . The difference between symmetrized and unsymmetrized solution is at most a couple of MeV's over a wide range of rotational and radial excited states. With larger quark masses this difference decreases.

In the general case we have two unknown matrices and the equations are much more complicated. In addition, a good initial guess for $v_{\perp 1}$ and $v_{\perp 2}$ is not easy to find. Usually, for the initial guess for $v_{\perp i}$ in the case of $l > 0$ (the $l = 0$ solution is trivial since $v_{\perp 1} = v_{\perp 2} = 0$) we use symmetrized equal-mass guess for $m = m_i$ and the same angular momentum quantum number l . Despite these complications, for $N \leq 15$ basis states we have found that our procedure converges very quickly no matter how large the difference between m_2 and m_1 is (i.e., no matter how bad initial guess is). In comparing with experimental data, we require only the lowest one or two eigenvalues, and these are determined within 10 MeV if we use 5, and within 1 MeV if we use 10 basis states. Fits with $N = 10$ are reliable and completely adequate for our purposes.

In Figs. 2 and 3 we compare the ground state quantum solution (for continuous values of l [4]) with the classical yrast solution. We show in these figures the predicted shifted Regge slope. The correspondence limit at large angular momentum is well satisfied.

5 Comparison with Experiment

As mentioned earlier, mesons containing at least one heavy quark will have relatively small dependences on quark spins. It seems realistic then to compare our predictions, which do not include quark spin, directly to spin averaged energy levels. There is however an additional feature which must be incorporated in the model to have a phenomenologically successful result.

5.1 Short range interaction

The flux tube configuration has been assumed to dominate when the quarks are widely separated. At short distance there must be an attractive singular interaction due to the single gluon exchange. We parametrize short and intermediate distance interaction by the usual potential

$$V_S(r) = -\frac{\kappa}{r} , \quad (58)$$

where at short distances $\kappa = \frac{4}{3}\alpha_s$. In the static limit the total potential then reduces to the well known “Cornell potential” [14]; a superposition of linear confinement $V_{conf} = ar$ and the Coulombic term (58). From general consideration we expect κ to “run” such that it will increase slowly as the quarks are separated (and hence the QCD scale decreases). In our comparison with data we will allow different κ ’s for $c\bar{c}$, $b\bar{b}$ and the heavy-light mesons.

5.2 Constant term

A constant term is also added yielding our model Hamiltonian

$$H = H_{RFT} + C - \frac{\kappa}{r}, \quad (59)$$

with H_{RFT} given by (53). The presence of the constant term does not change the model predictions but it does determine how the heavy quark masses change with QCD scale.

5.3 The Data and Comparison with Theory

We have used the quark model spectroscopic assignments and level masses of the *Particle Data Group* [15] to extract spin averaged states for heavy-light mesons (Table 1) and heavy-heavy mesons below flavor threshold (Table 2).

From QCD one expects that the strong coupling constant and quark masses will become smaller as the QCD scale increases [16]. Since the energy scale for mesons is in the non-perturbative regime, one cannot expect that perturbative expressions for the running of α_s and quark masses agree quantitatively with the model results. However, one can expect qualitative agreement.

In order to show this, we have performed series of fits using the generalized RFT model equations, in which we have fixed the constant C in the Hamiltonian, the string tension a , and also the light quark mass $m_{u,d}$. The free parameters of the fits were $\kappa(hl)$, $m_s(hl)$, $m_c(hl)$ and $m_b(hl)$ for the heavy-light mesons, $\kappa(c\bar{c})$ and $m_c(c\bar{c})$ for the $c\bar{c}$ mesons, and $\kappa(b\bar{b})$ and $m_b(b\bar{b})$ for the $b\bar{b}$ mesons. As an illustration of the quality of our fits, in table 1 and 2 we present fits for the heavy-heavy and heavy-light mesons for $C = -0.8GeV$, $a = 0.2GeV^2$ and $m_{u,d} = 0.3GeV$. The resulting

values of free fit parameters were

$$\begin{aligned}
& \left. \begin{aligned} m_s &= 0.487 \text{ GeV} \\ m_c &= 1.955 \text{ GeV} \\ m_b &= 5.391 \text{ GeV} \\ \kappa(hl) &= 0.576 \end{aligned} \right] \text{ hl mesons ,} \\
& \left. \begin{aligned} m_c &= 1.816 \text{ GeV} \\ \kappa(c\bar{c}) &= 0.532 \end{aligned} \right] c\bar{c} \text{ mesons ,} \\
& \left. \begin{aligned} m_b &= 5.153 \text{ GeV} \\ \kappa(b\bar{b}) &= 0.440 \end{aligned} \right] b\bar{b} \text{ mesons .}
\end{aligned} \tag{60}$$

As one can see from tables 1 and 2, fits are quite reasonable. All spin averaged heavy-light states were reproduced to within $2MeV$, while heavy-heavy states had errors within $10MeV$.

To observe running in the RFT model, let us first consider strong coupling constant $\kappa = \frac{4}{3}\alpha_s$. If we make the reasonable assumption that meson dynamics is governed roughly by the reduced mass of its constituent quarks, we expect that κ will decrease as we increase light quark mass ($m_{u,d}, m_s$), and therefore the reduced mass. Furthermore, as we go to from the heavy-light to $c\bar{c}$ and then to $b\bar{b}$ mesons, energy scale again increases, and coupling constant should decrease. This behaviour is exactly what we see in the Fig. 5.

As far as m_c and m_b are concerned, first of all we expect larger heavy quark masses for $m_{u,d} = 0$, since meson mass is roughly the sum of masses of its constituents, and this is shown on Figs. 6 and 7 (two dashed lines). Furthermore, the constant C should just renormalize the heavy quark masses in the heavy-heavy systems ($m_i \rightarrow m_i + \frac{1}{2}C$). From our fits we find (full lines on Figs. 6 and 7)

$$m_c(C) = 1.368GeV - 0.560C \quad (c\bar{c} \text{ mesons}) , \tag{61}$$

$$m_b(C) = 4.739GeV - 0.520C \quad (b\bar{b} \text{ mesons}) , \tag{62}$$

as expected. For the heavy-light systems, one might expect that we have $m_i \rightarrow m_i + C$, since we have kept the light quark masses fixed. For $m_{u,d} = 0.3\text{GeV}$ our fits give (dashed lines on Figs. 6 and 7)

$$m_c(C) = 1.078\text{GeV} - 1.094C , \quad (63)$$

$$m_b(C) = 4.571\text{GeV} - 1.025C , \quad (64)$$

and for $m_{u,d} = 0$ we find

$$m_c(C) = 1.188\text{GeV} - 1.081C , \quad (65)$$

$$m_b(C) = 4.660\text{GeV} - 1.024C . \quad (66)$$

Also, as stated before, one would expect that heavy quark masses should decrease as we go from the heavy-light to the heavy-heavy systems. As we can see from the Figs. 6 and 7, this is satisfied only for large negative values of C . This gives us an estimate of the upper limit for its possible values. In particular, C must be more negative than about -0.5GeV for both m_c and m_b to run in generally accepted manner.

Finally, from the Fig. 8 one can see that strange quark mass (obtained from the heavy-light fits) is almost independent of C .

6 Conclusions

In this paper we have discussed the relativistic flux tube model with arbitrary mass spinless quarks on its ends. We have obtained the classical rotational solution (yrast). The quantum solution is also proposed and solved numerically. In both the classical and quantized solutions an important ingredient is the condition of fixed center of momentum. The quantum solution also involved a careful treatment of symmetrization of the equations of motion. We have confined our comparison of the RFT with data to mesons containing at least one heavy quark primarily because

in this case spin dependence should be small enough that the spin averaged data can be compared to our spinless quark calculation. We have done a simultaneous fit to all of the available heavy-light and heavy-heavy data, and we have obtained an excellent fit to the spin averaged levels. To achieve this agreement we have allowed the short range coupling constant κ and the heavy quark masses m_c and m_b to run with QCD scale. We note that κ always decreases with increasing scale (reduced mass), and that the heavy quark masses decrease with increasing scale if the constant C is more negative than roughly -0.5GeV .

It will be important to include spin in our model. Fortunately the orbital angular momentum discussion in this paper will form a key ingredient of the more complete calculation. The results will be somewhat different even for the spin averaged case since a Darwin type term will arise due to the fermionic nature of the quarks both at long and short range. The RFT model formulation with fermionic quarks has been discussed in some detail [3] previously.

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References

- [1] Dan LaCourse and M. G. Olsson, Phys. Rev. D **39**, 2751 (1989).
- [2] Collin Olson, M. G. Olsson and Ken Williams, Phys. Rev. D **45**, 4307 (1992).
- [3] M. G. Olsson and Ken Williams, Phys. Rev. D **48**, 417 (1993).
- [4] C. Olson, M. G. Olsson and D. LaCourse, Phys. Rev. D **49**, 4675 (1994).
- [5] E. Eichten and F. Feinberg, Phys. Rev. D **23**, 2724 (1981); D. Gromes, Z. Phys. C **22**, 265 (1984); **26**, 401 (1984); A. Barchielli, E. Montaldi and G. M. Prosperi, Nucl. Phys. **B296**, 625 (1988); **B303**, 752(E) (1988).
- [6] A. Barchielli, N. Brambilla and G. M. Prosperi, Nuovo Cim **103A**, 59 (1989).
- [7] N. Brambilla and G. M. Prosperi, Phys. Lett. **236B**, 69 (1990).
- [8] W. Buchmüller, Phys. Lett. **112B**, 479 (1982).
- [9] Robert D. Pisarski and John D. Stack, Nucl. Phys. **286B**, 657 (1987).
- [10] A. Yu. Dubin, A. B. Kaidalov and Yu. A. Simonov, Journal of Nuclear Physics, **56/12**, 213 (1993); Phys. Lett. **323B**, 41 (1994).
- [11] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).
- [12] Alan Chodos and Charles B. Thorn, Nucl. Phys. **B72**, 509 (1974); M. Ida, Prog. Theor. Phys. **59**, 1661 (1978).
- [13] A discussion of the Galerkin method and the definition of the specific basis states used here is found in S. Jacobs, M. G. Olsson and C. J. Suchyta III, Phys. Rev. D **33**, 3338 (1986).

- [14] T. Appelquist, R. M. Barnett and K. D. Lane, *Ann. Rev. Nuc. Part. Sci.* **28**, 38 (1978).
- [15] Particle Data Group, *Phys. Rev. D* **50**, 1320 (1994).
- [16] H. Georgi and H. D. Politzer, *Phys. Rev. D* **14**, 1829 (1976).

TABLES

Table 1: Heavy-light spin averaged states. Spin-averaged masses are calculated in the usual way, by taking $\frac{3}{4}$ of the triplet and $\frac{1}{4}$ of the singlet mass.

state	spectroscopic label		spin-averaged	theory	error
	J^P	$^{2S+1}L_J$	mass (MeV)	(MeV)	(MeV)
<u>$c\bar{u}, c\bar{d}$ quarks</u>					
D (1867)	0^-	1S_0	$1S$ (1974)	1972	-2
D^* (2009)	1^-	3S_1			
D_1 (2424)	1^+	1P_1			
<u>$c\bar{s}$ quarks</u>					
D_s (1969)	0^-	1S_0	$1S$ (2075)	2077	2
D_s^* (2110)	1^-	3S_1			
D_{s_1} (2537)	1^+	1P_1			
<u>$b\bar{u}, b\bar{d}$ quarks</u>					
B (5279)	0^-	1S_0	$1S$ (5312)	5313	1
B^* (5325)	1^-	3S_1			
<u>$b\bar{s}$ quarks</u>					
B_s (5368)	0^-	1S_0	$1S$ (5409)	5408	-1

Table 2: Heavy-heavy spin averaged states. Since η_b 's have not been observed yet, in calculating spin averaged mass for all S states in $b\bar{b}$ system we assume $\Upsilon - \eta_b = 40 \pm 20 \text{ MeV}$ on the grounds that this splitting should be approximately one third of the corresponding splitting in the $c\bar{c}$ systems. We estimate that the error introduced in this way is probably less than 5 MeV . We also assume that p -wave hyperfine splitting in the $b\bar{b}$ mesons is negligible.

state	spectroscopic label		spin-averaged	theory	error
	J^P	$^{2S+1}L_J$	mass (MeV)	(MeV)	(MeV)
<u>$c\bar{c}$ quarks</u>					
η_c (2979)	0^-	1S_0	$1S$ (3068)	3066	-2
ψ (3097)	1^-	3S_1			
χ_{0c} (3415)	0^+	3P_0			
χ_{1c} (3511)	1^+	3P_1	$1P$ (3525)	3519	-6
χ_{2c} (3556)	2^+	3P_2			
h_c (3525)	1^+	1P_1			
η'_c (3594)	0^-	2^1S_0	$2S$ (3663)	3672	9
ψ' (3686)	1^-	2^3S_1			
<u>$b\bar{b}$ quarks</u>					
Υ (9460)	1^-	3S_1	$1S$ (9450)	9451	1
Υ' (10023)	1^-	2^3S_1	$2S$ (10013)	10004	-9
Υ'' (10355)	1^-	3^3S_1	$3S$ (10345)	10349	4
χ_{0b} (9860)	0^+	3P_0	$1P$ (9900)	9905	5
χ_{1b} (9892)	1^+	3P_1			
χ_{2b} (9913)	2^+	3P_2			
χ'_{0b} (10232)	0^+	2^3P_0	$2P$ (10261)	10260	-1
χ'_{1b} (10255)	1^+	2^3P_1			
χ'_{2b} (10268)	2^+	2^3P_2			

FIGURES

Figure 1: Portion of a meson consisting of a segment of flux tube from the center of momentum to the i^{th} quark.

Figure 2: Yrast solution (solid curve) compared to the quantum solution (dashed curve) for the displaced Regge slope with $m_1 = 0$ and $m_2 = 1.5\text{GeV}$. We compare our solution for the same value of J (yrast) and l (quantum). In both cases the transition from heavy-light to light-light dynamics is clearly seen by the slope dropping from two to unity.

Figure 3: Yrast solution (solid curve) compared to the quantum solution (dashed curve) for the displaced Regge slope with $m_1 = 0.5\text{GeV}$ and $m_2 = 1.5\text{GeV}$. In this case we observe the transition from heavy-heavy dynamics at low l to light-light dynamics at large l .

Figure 4: Comparison of the symmetrized and non-symmetrized equations for quark masses $m_1 = m_2 = 0$ with $N = 50$ basis states. The non-symmetrized equations (32) and (33) discussed in [4] are found to agree with the fully symmetrized solutions of (39) and (40) to better than 5MeV .

Figure 5: Strong coupling constant as a function of C in the heavy-light (for $m_{u,d} = 0$ and $m_{u,d} = 0.3\text{GeV}$) $c\bar{c}$ and $b\bar{b}$ mesons. The values of κ found are roughly independent of C , but systematically fall with running QCD scale (reduced mass).

Figure 6: Mass of the c quark as a function of C in the heavy-light (for $m_{u,d} = 0$ and $m_{u,d} = 0.3\text{GeV}$, dashed lines) and $c\bar{c}$ mesons. For C less than about -0.5GeV m_c falls with increasing QCD scale (reduced mass).

Figure 7: Mass of the b quark as a function of C in the heavy-light (for $m_{u,d} = 0$ and $m_{u,d} = 0.3\text{GeV}$, dashed lines) and $b\bar{b}$ mesons. For C less than about -0.3GeV m_c falls with increasing QCD scale (reduced mass).

Figure 8: Mass of the s quark as a function of C in the heavy-light (for $m_{u,d} = 0$ and $m_{u,d} = 0.3\text{GeV}$) mesons.

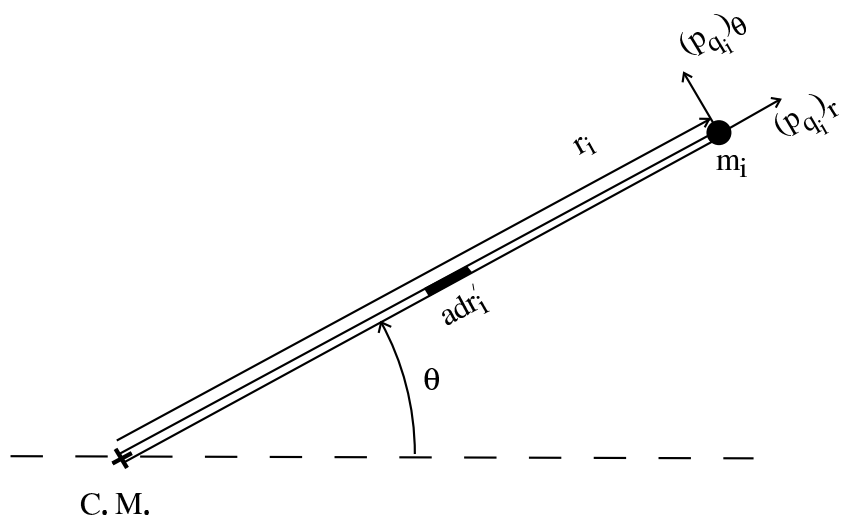


Figure 1

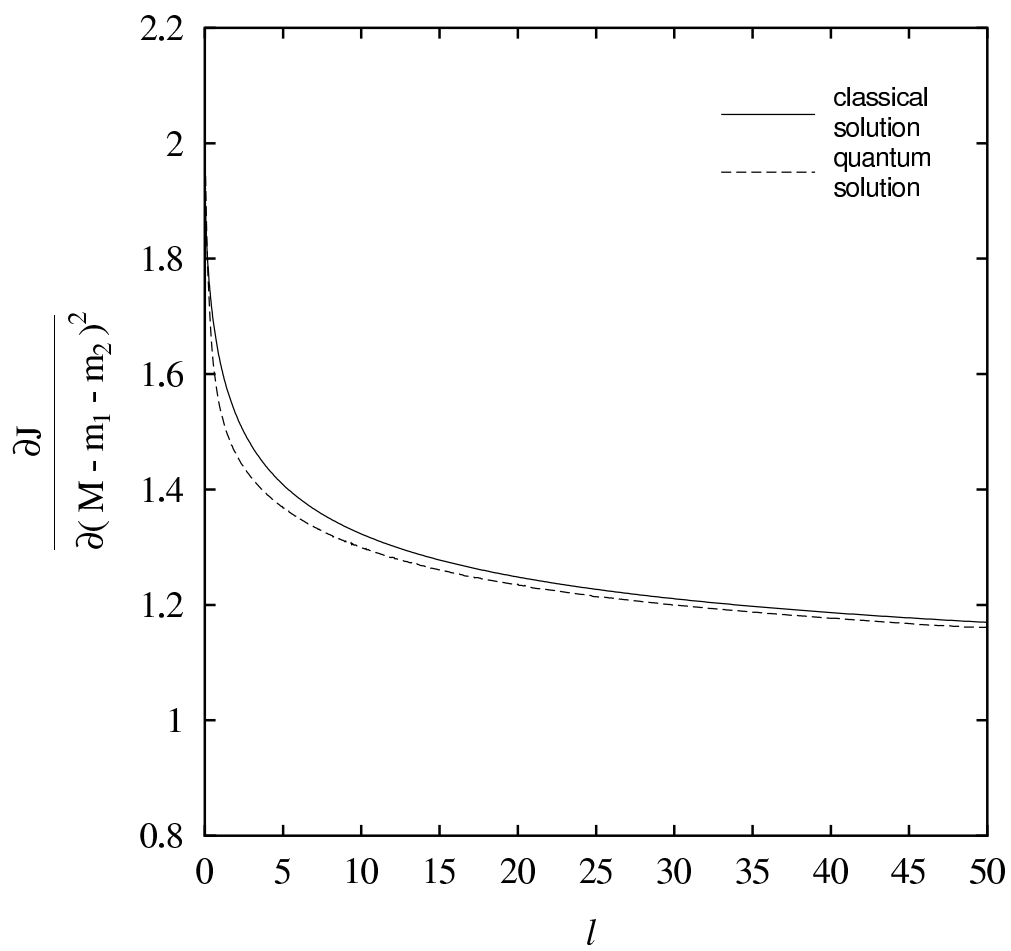


Figure 2

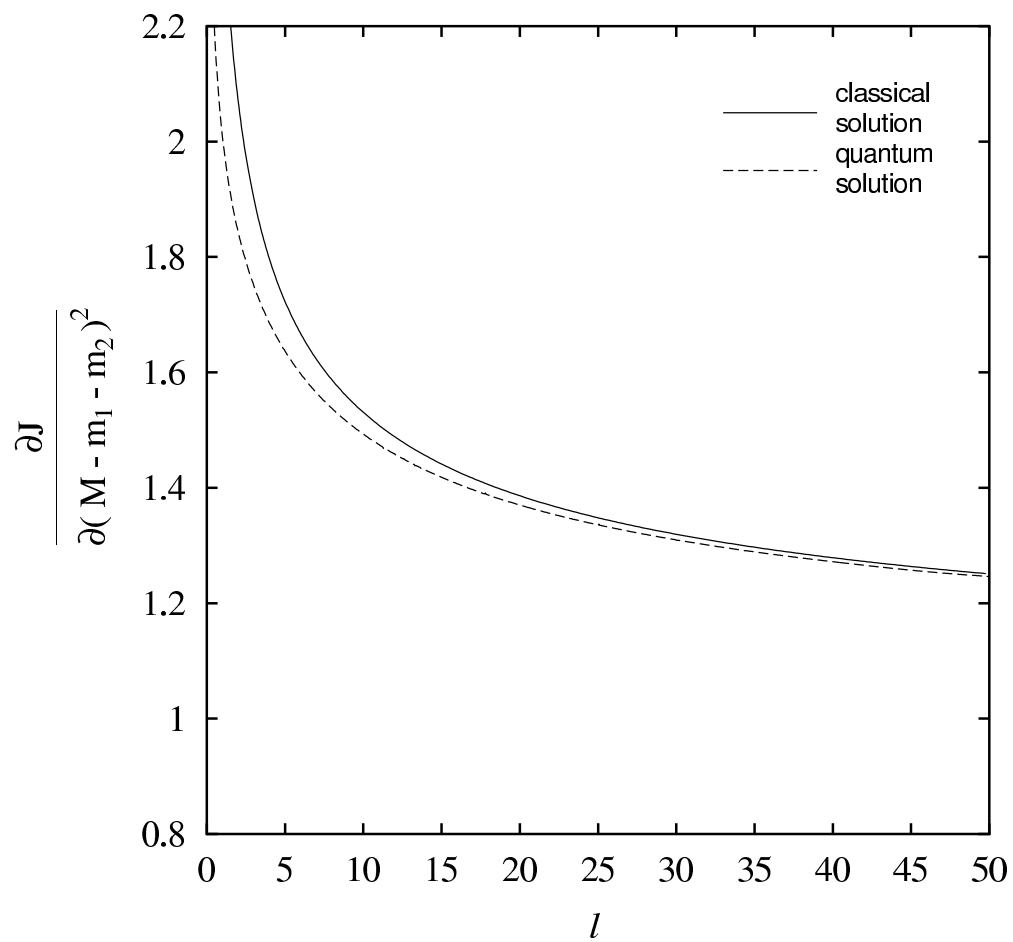


Figure 3

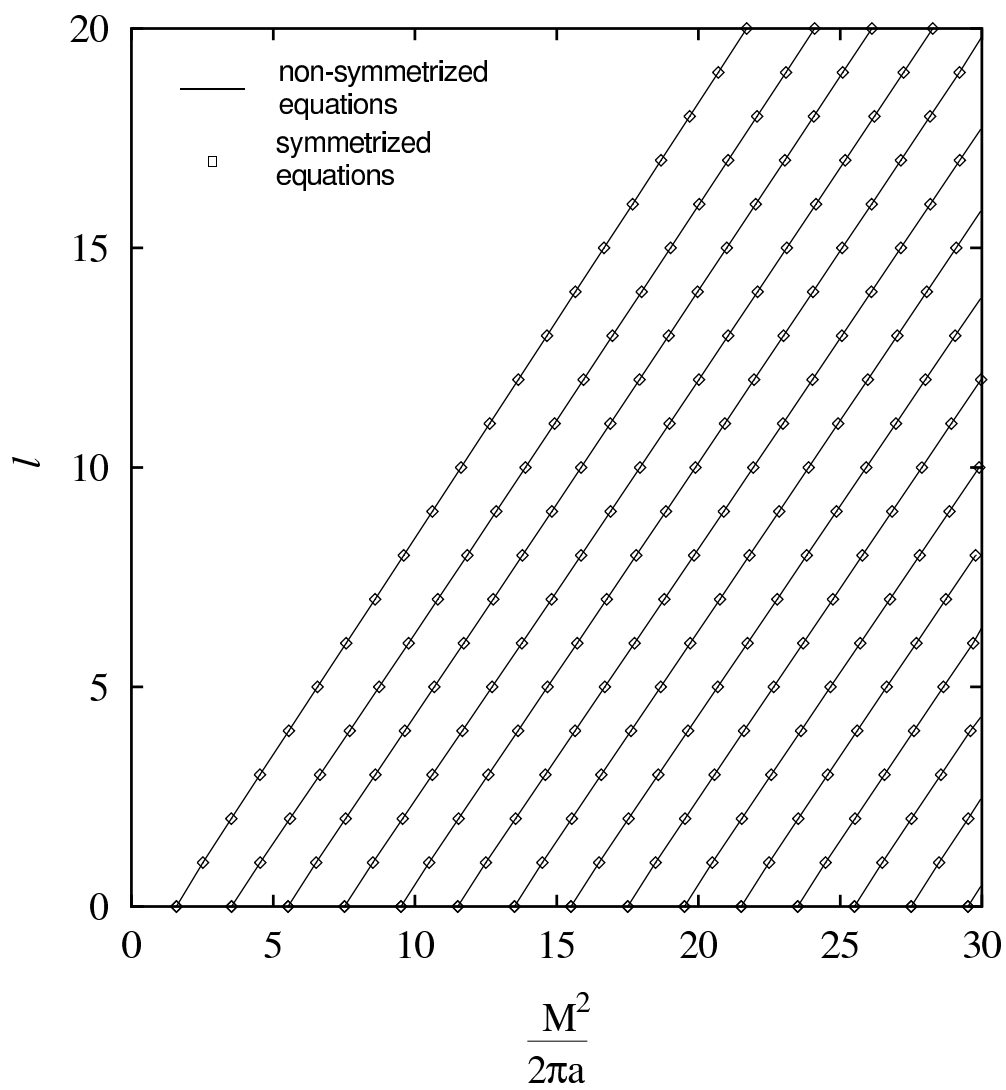


Figure 4

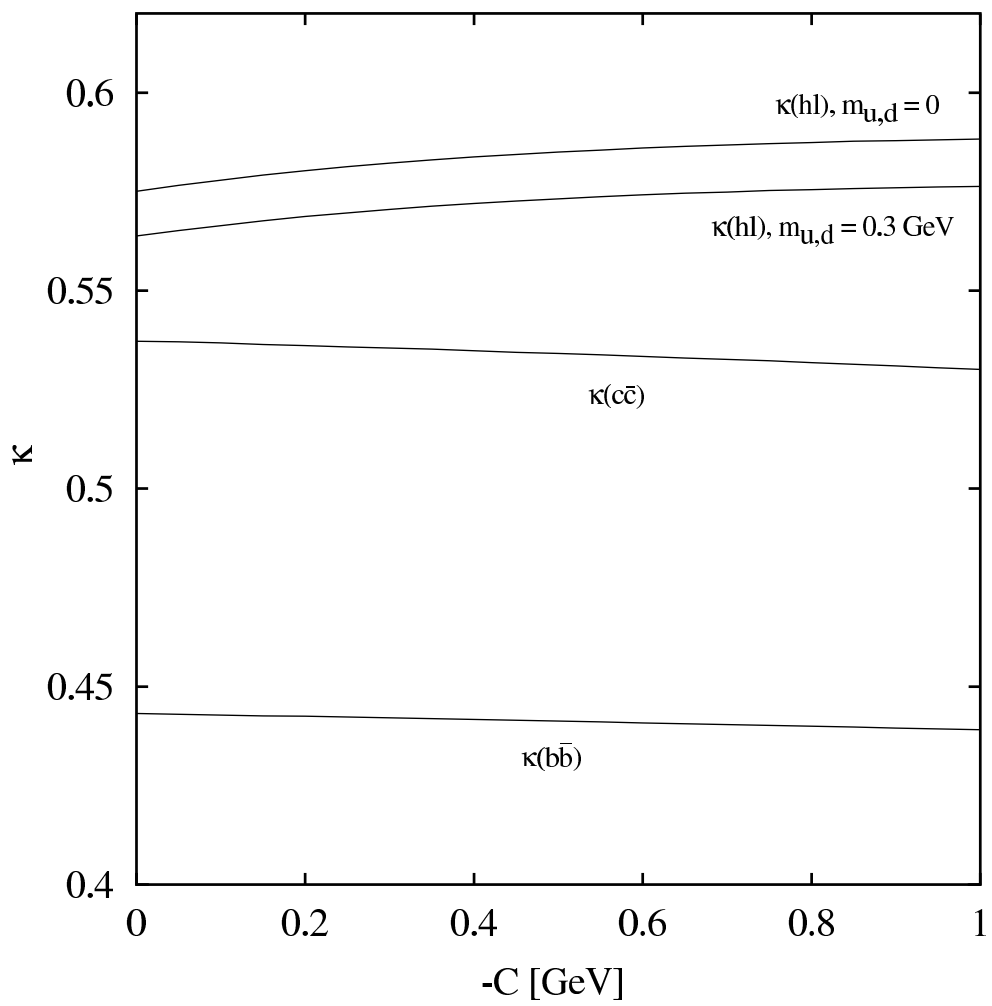


Figure 5

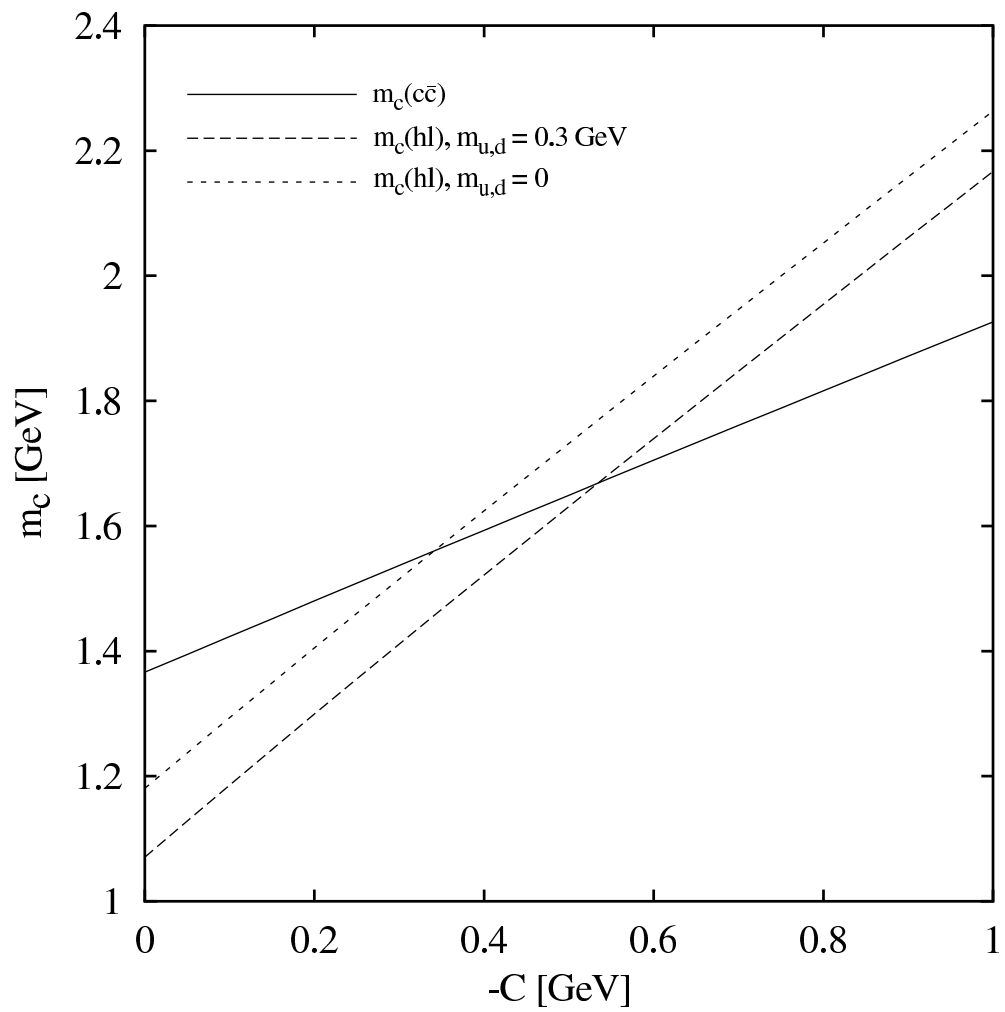


Figure 6

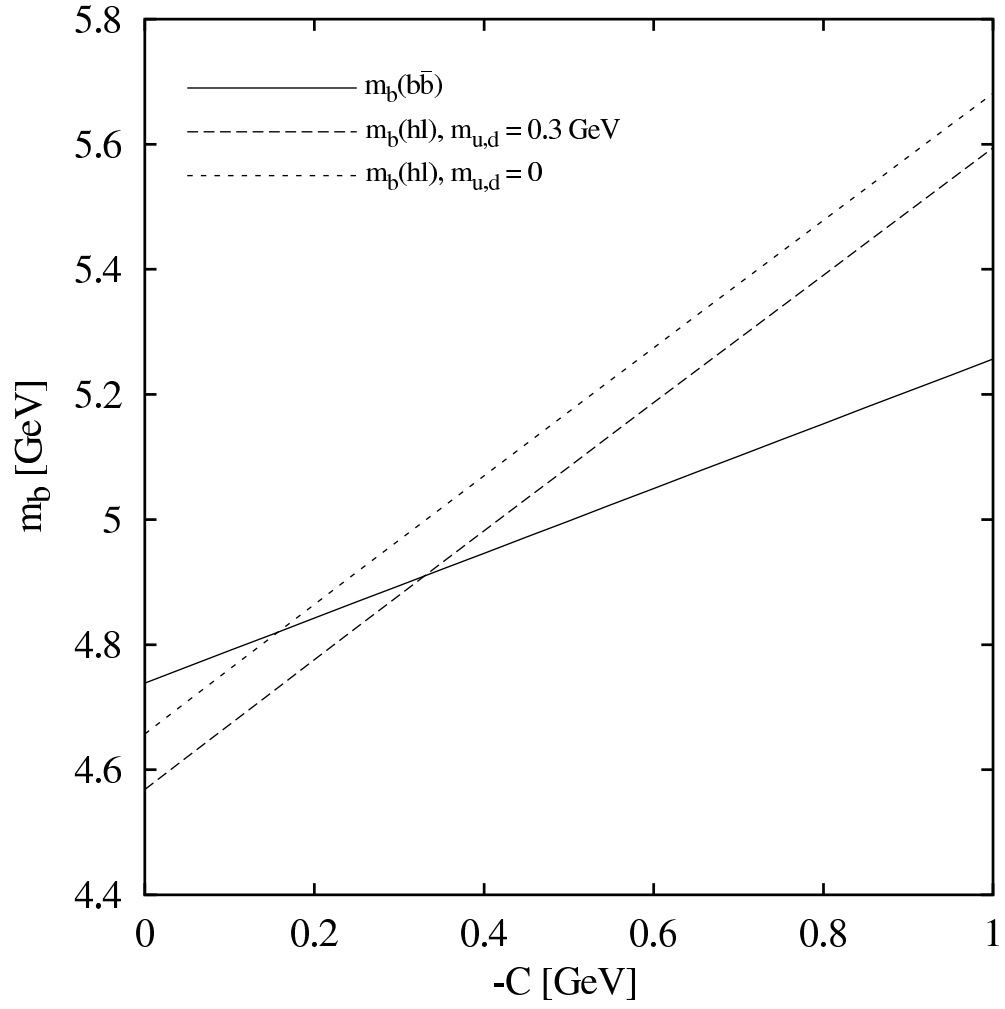


Figure 7

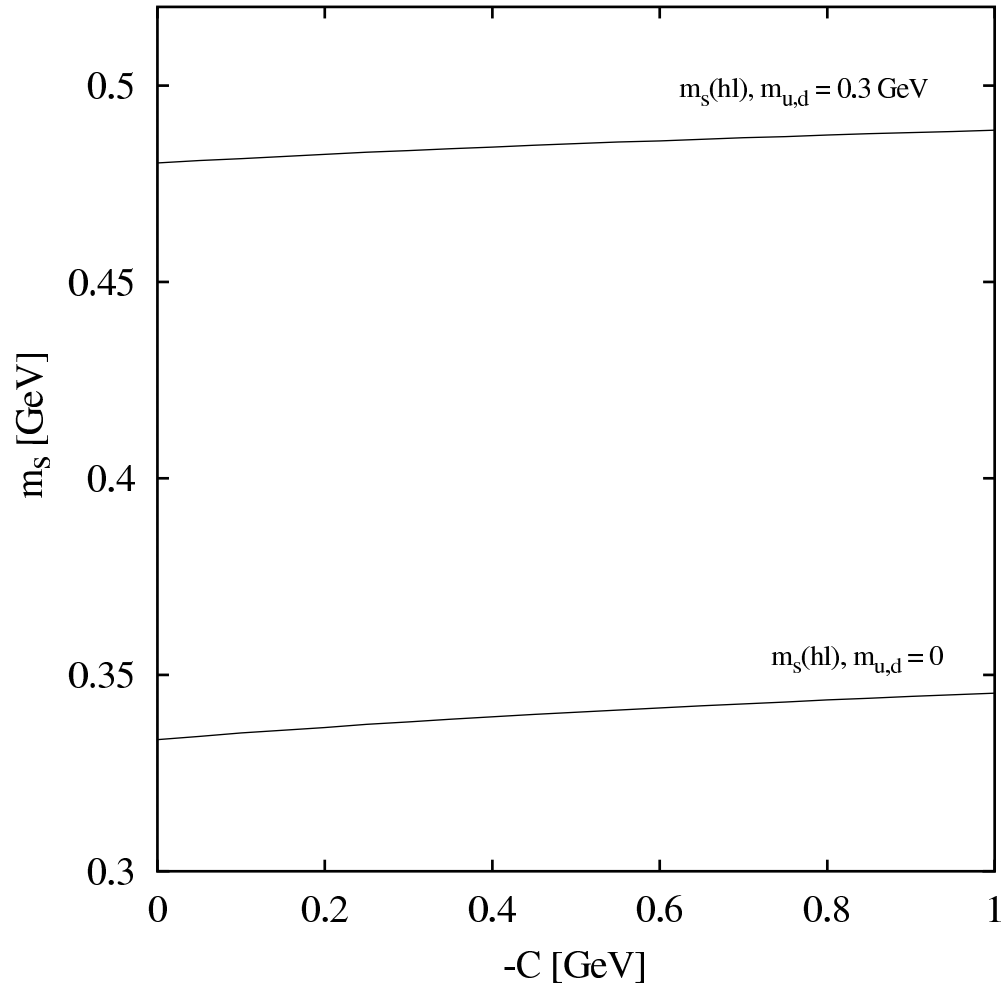


Figure 8